

On Logic of Formal Provability and Explicit Proofs

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Abstract

In 1933, Gödel considered two modal approaches to describing provability. One captured formal provability and resulted in the logic GL and Solovay's Completeness Theorem. The other was based on the modal logic S4 and led to Artemov's Logic of Proofs LP. In this paper, we study introduced by the author logic GLA, which is a fusion of GL and LP in the union of their languages. GLA is supplied with a Kripke-style semantics and the corresponding completeness theorem. Soundness and completeness of GLA with respect to the arithmetical provability semantics is established.

1 Introduction

Gödel in [11] suggested a provability reading of modal logic **S4**, which is axiomatized over the classical logic by the following list of postulates:

$$\begin{array}{ll} \Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G) & \text{Deductive Closure/Normality} \\ \Box F \rightarrow \Box \Box F & \text{Positive Introspection/Transitivity} \\ \Box F \rightarrow F & \text{Reflection} \end{array}$$

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and the *Necessitation Rule*: $\vdash F \Rightarrow \vdash \Box F$.

Gödel considered the interpretation of $\Box F$ as the formal provability predicate

$$F \text{ is provable in Peano Arithmetic PA}$$

and noticed that this semantics is inconsistent with **S4**.

Indeed, $\Box(\Box F \rightarrow F)$ can be derived in **S4**. On the other hand, interpreting \Box as the predicate “*Provable*” of formal provability in Peano Arithmetic PA and F as *false* \perp , converts this formula into the false statement that the consistency of PA is internally provable in PA:

$$\text{Provable}(\text{Consis PA}).$$

1.1 Formal provability spills over to non-standard proofs

Let $\text{Proof}(x, F)$ be a standard proof predicate (cf. [4, 8, 9]) x is a proof for F ; $\text{Provable } F$ be $\exists x \text{Proof}(x, F)$.

Peano Arithmetic PA cannot distinguish between standard and nonstandard numbers; given $\exists x \text{Proof}(x, F)$, x may be a nonstandard number, hence not a code of any derivation in PA. It means that $\text{Provable } F \rightarrow F$ can fail in a model, and hence is not derivable in PA.

Indeed, consider a theory $\mathsf{T} = \text{PA} + \text{Provable } \perp$. T is consistent, since PA does not prove $\neg \text{Provable } \perp$. Hence T has a model M in which $\text{Provable } \perp$ holds, but \perp does not.

So, the formal provability interpretation of **S4** does not work; a provability calculus was left without a semantics and a provability semantics was left without a calculus thus opening two problems:

1. Find a precise provability semantics for **S4**;
2. Find a modal logic of formal provability *Provable*.

Problem 2 was solved in 1976 by Solovay [24], who proved the completeness of Gödel-Löb logic GL with respect to the formal provability in arithmetic PA.

In 1995, Problem 1 found its solution in Artemov’s Logic of Proofs LP which provided a semantics of explicit proofs for **S4** ([2, 3]).

1.2 Gödel-Löb logic of formal provability

Logic of Formal Provability GL (standing for Gödel-Löb) is given by the following list of postulates:

1. *Axioms and rules of classical propositional logic*
2. $\Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G)$ *Deductive Closure/Normality*
3. $\Box F \rightarrow \Box \Box F$ *Verification/Transitivity*
4. $\Box(\Box F \rightarrow F) \rightarrow \Box F$ *Löb Axiom*
5. *Necessitation Rule:*
$$\frac{\vdash F}{\vdash \Box F}$$

Formal provability interpretation of a modal language is a mapping $*$ from the set of modal formulas to the set of arithmetical sentences such that $*$ agrees with Boolean connectives and constants and

$$(\Box G)^* = \text{Provable } G^*.$$

Solovay's completeness theorem ([8, 24]):

$$\text{GL} \vdash F \quad \text{iff} \quad \text{for all formal provability interpretations } *, \text{ PA} \vdash F^*.$$

In 1938, Gödel outlined a way to provide a provability semantics for S4 ([12]): modality there should be read explicitly as proof assertions $t:F$ interpreted as

$$t \text{ is a proof of } F \text{ in Peano Arithmetic PA}.$$

This Gödel's suggestion was realized in Artemov's Logic of Proofs ([2, 3]).

1.3 Artemov's Logic of Proofs

Proof terms in LP are built from constants and variables by two binary operations *application* “.” and *sum* “+”, and one unary operation *proof checker* “!”. Formulas of LP are built as the usual propositional formulas with an additional formation rule: whenever F is a formula and t a proof terms, $t:F$ is a formula.

Axioms and rules of the Logic of Proofs LP are those of classical propositional logic plus axioms

$$\begin{array}{ll}
s:(F \rightarrow G) \rightarrow (t:F \rightarrow [s \cdot t]:G) & \text{Application} \\
t:F \rightarrow !t:(t:F) & \text{Proof Checker} \\
s:F \rightarrow [s+t]:F, \quad t:F \rightarrow [s+t]:F & \text{Sum} \\
t:F \rightarrow F & \text{Explicit Reflection}
\end{array}$$

Each axiom A is assumed internally provable, which is represented by formula $c:A$ where c is a proof constant. The fundamental property of LP is given by Artemov's Realization Theorem ([2, 3]): *for each theorem F of S4 one could recover a witness (proof term) to each occurrence of \Box in F in such a way that the resulting formula F^r is derivable in LP*. This theorem embeds S4 into LP. Further interpretation of LP proof terms as formal proofs in PA ([2, 3]) provided a Gödelian provability semantics for LP and S4 and completed Gödel's project of 1933. Nowadays, the Logic of Proofs has evolved into a general logical theory of justification [5, 6, 7].

1.4 Comparing two Gödel approaches to provability

Logic of formal provability GL formalizes Gödel's second incompleteness theorem

$$\neg\Box(\neg\Box\perp),$$

Löb's theorem

$$\Box(\Box F \rightarrow F) \rightarrow \Box F,$$

and a number of other meaningful provability principles.

Logic of Proofs LP represents proofs explicitly, naturally extends typed λ -calculus, modal logic, and modal λ -calculus.

GL and S4/LP complement each other by addressing different areas of application. GL finds applications in traditional proof theory. LP targets areas of mathematical theories of knowledge and justification, foundations of verification, typed theories and lambda-calculi, etc.

1.5 Mixture of provability and explicit proofs

Certain principles require a mixture of both provability and explicit proofs. Consider the negative introspection principle. Its purely modal formulation

$\neg\Box F \rightarrow \Box\neg\Box F$ is not valid as a provability principle. Indeed, let F be \perp . Then $\neg\Box\perp$ reads as *Consis PA* and the whole formula as

$$\text{Consis PA} \rightarrow \text{Provable}(\text{Consis PA}),$$

which is false, by Gödel's Second Incompleteness Theorem.

There is no explicit negative introspection either. The principle $\neg p:S \rightarrow t:(\neg p:S)$, where p and t are proof terms and S is a propositional variable, is not valid. Indeed, fix an interpretation $*$ of p and t and the standard Gödel proof predicate. There are infinitely many arithmetical instances of S for which the antecedent holds. Hence t^* should be a proof of infinitely many theorems, which is impossible. However, the mixed language of proofs and provability fits this version of negative introspection:

$$\neg p:F \rightarrow \Box(\neg p:F)$$

is arithmetically provable, by Σ -completeness of PA, according to which for each Σ -formula σ ,

$$\text{PA} \vdash \sigma \rightarrow \text{Provable } \sigma.$$

We develop introduced in [18] a joint logic of formal provability and explicit proofs GLA (Gödel-Löb-Artëmov logic) in the language with provability assertions $\Box F$ and proof assertions $t:F$, find Kripke semantics for GLA and establish the arithmetical completeness of this logic.

GLA proved to be useful for applications in formal epistemology where it became a template for a family of epistemic logics with justifications (cf. [6, 7]). An elaborate proof theory of GLA and another version of Kripke models for GLA were offered by Kurokawa in [16, 17].

2 Description and basic properties of GLA

The following two systems are predecessors of GLA:

- system B from [1], which does not have operations on proofs;
- system LPP from [23, 25] in an extension of languages of the logic of formal provability GL and the Logic of Proofs LP.

Immediate successors of GLA are the logic GrzA of strong provability and explicit proofs [20], and symmetric logic of proofs and provability [21].

Language of GLA.

Proof terms are built from *proof variables* x, y, z, \dots and *proof constants* a, b, c, \dots by means of two binary operations: *application* ‘ \cdot ’ and *union* ‘ $+$ ’, and one unary *proof checker* ‘ $!$ ’.

Formulas of GLA are defined by the grammar

$$A = S \mid A \rightarrow A \mid A \wedge A \mid A \vee A \mid \neg A \mid \Box A \mid t:A ,$$

where t stands for any proof term and S for any sentence letter.

Axioms and rules of both Gödel-Löb logic GL and LP, together with three specific principles connecting explicit proofs with formal provability, constitute GLA_\emptyset .

I. Axioms of classical propositional logic

Standard axioms of the classical logic (e.g., A1-A10 from [15])

II. Axioms of Provability Logic GL

GL1	$\Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G)$	<i>Deductive Closure/Normality</i>
GL2	$\Box F \rightarrow \Box \Box F$	<i>Positive Introspection/Transitivity</i>
GL3	$\Box(\Box F \rightarrow F) \rightarrow \Box F$	<i>Löb Principle</i>

III. Axioms of the Logic of Proofs LP

LP1	$s:(F \rightarrow G) \rightarrow (t:F \rightarrow [s \cdot t]:G)$	<i>Application</i>
LP2	$t:F \rightarrow !t:(t:F)$	<i>Proof Checker</i>
LP3	$s:F \rightarrow [s+t]:F, \quad t:F \rightarrow [s+t]:F$	<i>Sum</i>
LP4	$t:F \rightarrow F$	<i>Explicit Reflection</i>

IV. Axioms connecting explicit and formal provability

C1	$t:F \rightarrow \Box F$	<i>Explicit-Implicit connection</i>
C2	$\neg t:F \rightarrow \Box \neg t:F$	<i>Explicit-Implicit Negative Introspection</i>
C3	$t:\Box F \rightarrow F$	<i>Explicit-Implicit Reflection</i>

V. Rules of inference

R1	$F \rightarrow G, F \vdash G$	<i>Modus Ponens</i>
R2	$\vdash F \Rightarrow \vdash \Box F$	<i>Necessitation</i>
R3	$\vdash \Box F \Rightarrow \vdash F$	<i>Reflection Rule</i>

A **Constant Specification CS** for GLA is the set of formulas

$$\{c_1:A_1, c_2:A_2, c_3:A_3, \dots\},$$

where each A_i is an axiom of GLA_\emptyset and each c_i is a proof constant.

$$\text{GLA}_{CS} = \text{GLA}_\emptyset + CS,$$

$$\text{GLA} = \text{GLA}_{CS} \text{ with the "total" } CS.$$

Theorem 1 (Internalization Theorem).

If $\text{GLA} \vdash F$ then for some proof term p , $\text{GLA} \vdash p.F$.

Proof. Induction on a derivation of F .

Base: F is an axiom. Then use Constant Specification. In this case, p is a proof constant.

Induction steps: by internalized rules of GLA .

Internalization of *Modus Ponens* immediately follows from the Application axiom **LP1**.

Internalization of Necessitation rule $\vdash F \Rightarrow \vdash \Box F$:

For each F there is $t(x)$ such that $\text{GLA} \vdash x.F \rightarrow t(x):\Box F$

1. $x.F \rightarrow \Box F$ - axiom Explicit-Implicit Connection **C1**;
2. $a.(x.F \rightarrow \Box F)$ - , from 1, by Constant Specification;
3. $x.F \rightarrow !x.x.F$ - axiom Proof Checker **LP2**;
4. $!x.x.F \rightarrow (a.!x):\Box F$ - from 2, by Application **LP1**;
5. $x.F \rightarrow (a.!x):\Box F$ - from 3,4, by propositional logic.

Now put $t(x) = a.!x$.

Internalization of Reflection rule $\vdash \Box F \Rightarrow \vdash F$

For each F there is $s(x)$ such that $\text{GLA} \vdash x:\Box F \rightarrow s(x):F$

1. $x:\Box F \rightarrow F$ - axiom Explicit-Implicit Reflection **C3**;
2. $b.(x:\Box F \rightarrow F)$ from 1, by Constant Specification;
3. $x:\Box F \rightarrow !x.x:\Box F$ - Proof Checker **LP2**;
4. $!x.x:\Box F \rightarrow (b.!x):F$ - from 2, by Application **LP1**;
5. $x:\Box F \rightarrow (b.!x):F$ - from 3,4, by propositional logic.

Now put $s(x) = b.!x$. Note that in 2, we need an *internalized* Explicit-Implicit Reflection!

□

The list of postulated axioms and rules of GLA contains some principles which are derivable from the rest of the system. Such redundancies are

generally acceptable to make exposition more readable. For example, in **GLA** (as well as in the Provability Logic **GL**) the positive introspection axiom **GL2** is derivable from the rest of the system (cf. [8]). In **GLA** the same holds for Reflection Axiom **LP4**, Necessitation Rule **R2** and Reflection Rule **R3**. In all these cases we decide to postulate the corresponding principles for the sake of more concise definitions of important subsystems of **GLA**.

Note that for any finite constant specification CS the rule of necessitation is not redundant in \mathbf{GLA}_{CS} since to emulate **R2** one needs an infinite constant specifications.

Here is an example of a yet more delicate dependency in **GLA**: even though Explicit-Implicit Reflection Axiom **C3** is derivable from the rest of \mathbf{GLA}_\emptyset (Proposition 1 below), proof constants corresponding to **C3** are needed to guarantee the Internalization Property of **GLA** (cf. Theorem 1). Hence, we keep **C3** as a basic postulate of **GLA**.

Proposition 1 ***C3** is derivable from the rest of \mathbf{GLA}_\emptyset .*

Proof. The following is a derivation of $t:\Box F \rightarrow F$ in \mathbf{GLA}_\emptyset without **C3**.

1. $\neg\Box F \rightarrow \neg t:\Box F$, contrapositive of **LP4**;
2. $\neg t:\Box F \rightarrow \Box(\neg t:\Box F)$, axiom **C2**;
3. $\Box(\neg t:\Box F) \rightarrow \Box(t:\Box F \rightarrow F)$, by reasoning in **GL**;
4. $\neg\Box F \rightarrow \Box(t:\Box F \rightarrow F)$, from 1,2, and 3;
5. $\Box F \rightarrow \Box(t:\Box F \rightarrow F)$, by reasoning in **GL**;
6. $\Box(t:\Box F \rightarrow F)$, from 4 and 5;
7. $t:\Box F \rightarrow F$, by **R3**.

□

GLA is closed under substitutions of proof terms for proof variables and formulas for propositional variables, enjoys the deduction theorem, and contains both **GL** and **LP**.

2.1 Some principles of **GLA**

Positive Introspection: $\mathbf{GLA} \vdash t:F \rightarrow \Box t:F$

1. $t:F \rightarrow !t:t:F$ - Proof Checker axiom **LP2**;
2. $!t:t:F \rightarrow \Box t:F$ - Explicit-Implicit Connection axiom **C1**;

3. $t:F \rightarrow \Box t:F$ - from 1,2, by propositional logic.

Stability of proof assertions: $\text{GLA} \vdash \Box t:F \vee \Box \neg t:F$

4. $\neg t:F \rightarrow \Box \neg t:F$ - Explicit-Implicit Negative Introspection **C2**;
5. $\Box t:F \vee \Box \neg t:F$ - from 3,4, by propositional logic.

Explicit version of Löb Principle. In $\Box(\Box F \rightarrow F) \rightarrow \Box F$ both modalities of the depth 1 can be read explicitly as

$$x:(\Box F \rightarrow F) \rightarrow l(x):F$$

for some proof term $l(x)$. Indeed,

1. $x:(\Box F \rightarrow F) \rightarrow t(x):\Box(\Box F \rightarrow F)$ - by Internalized Necessitation Rule;
2. $c:(\Box(\Box F \rightarrow F) \rightarrow \Box F)$ - from Löb Principle **GL3** by Constant Specification;
3. $t(x):\Box(\Box F \rightarrow F) \rightarrow (c \cdot t(x)):\Box F$ - from 1,2 by Application **LP1**;
4. $(c \cdot t(x)):\Box F \rightarrow s(c \cdot t(x)):F$ - by Internalized Reflection Rule;
5. $x:(\Box F \rightarrow F) \rightarrow s(c \cdot t(x)):F$ - from 1,3,4.

Löb Principle cannot be realized in full. Suppose for some proof polynomials u and v ,

$$\text{GLA} \vdash x:(u:\perp \rightarrow \perp) \rightarrow v:\perp,$$

hence $\text{GLA} \vdash x:(u:\perp \rightarrow \perp) \rightarrow \perp$ and so $F = \neg x:(u:\perp \rightarrow \perp)$ is derivable in **GLA**. Consider a **GLA**-derivable formula

$$G = c:(u:\perp \rightarrow \perp).$$

Let us perform a substitution $\tau = [c/x]$ to both F and G . Then F becomes $\neg c:(\tau u:\perp \rightarrow \perp)$ and G yields $c:(\tau u:\perp \rightarrow \perp)$, which is impossible.

2.2 Realizable provability principles.

A Franco Montagna's question which theorems of **GL** are realizable in **GLA**, has been answered by Evan Goris in [13, 14].

It follows from the realization theorem for **LP** that all formulas of $\text{GL} \cap \text{S4}$ are realizable in **LP**, and the question was actually whether proof terms of **GLA** were capable of realizing some other modal theorems of **GL**. Goris' Theorem yields that it is not the case.

Theorem [13, 14]. *Only those theorems of GL are realizable in GLA which are from S4.*

3 Models for GLA

In this section, we build Kripke-style models for GLA, which were described in [19].

A *frame* is a standard GL-frame $(W, \prec, root)$ with the root node $root$, where W is a non-empty set of *possible worlds*, \prec is a binary transitive and conversely well-founded *accessibility* relation on W (a relation \prec is conversely well-founded if any increasing chain $a_1 \prec a_2 \prec a_3 \prec \dots$ is finite).

Possible evidence relation (first considered by Mkrtychev and then by Fitting) is a relation \mathcal{E} between proof terms and formulas such that the following *closure conditions* are met:

Application: $\mathcal{E}(s, F \rightarrow G)$ and $\mathcal{E}(t, F)$ implies $\mathcal{E}(s \cdot t, G)$.

Proof Checker: $\mathcal{E}(t, F)$ implies $\mathcal{E}(!t, (t:F))$.

Sum: $\mathcal{E}(s, F)$ or $\mathcal{E}(t, F)$ implies $\mathcal{E}(s + t, F)$.

Model is a structure $\mathcal{M} = (W, \prec, root, \mathcal{E}, \Vdash)$; here \Vdash is a relation between worlds and formulas such that

1. \Vdash respects Boolean connectives at each world
($u \Vdash F \wedge G$ iff $u \Vdash F$ and $u \Vdash G$; $u \Vdash \neg F$ iff $u \not\Vdash F$, etc.);
2. $u \Vdash \Box F$ iff $v \Vdash F$ for every $v \in W$ with $u \prec v$;
3. $u \Vdash t:F$ iff $\mathcal{E}(t, F)$ and $v \Vdash F$ for every $v \in W$.

Following Solovay, we define

$$\mathcal{H}(F) = \{\Box G \rightarrow G \mid \Box G \text{ is a subformula of } F\};$$

for a set of formulas X ,

$$\mathcal{H}(X) = \bigcup_{F \in X} \mathcal{H}(F).$$

A model \mathcal{M} is called *F-sound* if $root \Vdash \mathcal{H}(F)$. For a set of formulas X , \mathcal{M} is *X-sound* if \mathcal{M} is *F-sound* for each $F \in X$.

For a given constant specification CS , a model \mathcal{M} is a CS -model if \mathcal{M} is CS -sound and CS holds in \mathcal{M} .

Theorem 2 (Soundness) *For any formula F and any constant specification CS , if F is derivable in GLA_{CS} then F holds in each F -sound CS -model.*

Theorem 3 (Completeness) *For any finite constant specification CS if F is not derivable in GLA_{CS} , then there is an F -sound CS -model with a finite frame where F does not hold.*

Proof goes by a canonical model construction with the use of technique developed by Solovay [24], Artemov [1], and Fitting [10]. GLA_\emptyset exhibits some sort of a finite model property, which also yields the decidability of GLA_{CS} for any given finite constant specification:

Theorem 4 *For any finite constant specification CS , the logic GLA_{CS} is decidable.*

4 Provability semantics for GLA, completeness

In what follows, all proof predicates are assumed *normal* ([3]), i.e., satisfying two properties.

1. Finiteness of proofs.

For every k set $T(k) = \{\varphi \mid \text{Proof}(k, \varphi)\}$ is finite, the function from k to $T(k)$ is computable.

2. Conjoinability of proofs.

For any k and l there is n such that

$$T(k) \cup T(l) \subseteq T(n).$$

Prime example: Gödel's proof predicate.

Arithmetical interpretation of GLA is the sum of the intended arithmetical interpretations for GL and LP. In particular,

$$\begin{aligned} (\Box G)^* &= \text{Provable } G^*; \\ (p:F)^* &= \text{Proof}(p^*, F^*). \end{aligned}$$

Theorem 5 (Soundness of GLA with respect to arithmetical provability)

For any Constant Specification CS and any arithmetical interpretation $$ respecting CS, if $\text{GLA}_{CS} \vdash F$ then $\text{PA} \vdash F^*$.*

Proof. It is immediate that Reflection Rule is valid: if *Provable F* is derivable in PA, then *Provable F* is true hence F is provable.

Validity of C1 and C2 immediately follows from Σ -completeness of PA.

Soundness of Explicit-Implicit Reflection takes place since $t:\Box F \rightarrow F$ is derivable from other principles of GLA, which is already proved sound. \square

Arithmetical completeness of GLA_\emptyset could be established following arithmetical completeness proofs from [1, 2, 3] (cf. also [25]).

Theorem 6 (Arithmetic completeness) *For any finite constant specification CS, if $\text{GLA}_{CS} \not\vdash F$, then there exists a CS-interpretation $*$ such that $\text{PA} \not\vdash F^*$.*

Proof. The claim of the theorem follows from the arithmetical completeness of GLA_\emptyset . \square

4.1 Explicit-Implicit Reflection vs. Implicit-Explicit Reflection

Explicit-Implicit Reflection $x:\Box F \rightarrow F$, as we have seen in Theorem 5, is arithmetically valid. However, the Implicit-Explicit Reflection

$$\text{IER} = \Box x:P \rightarrow P$$

is not a provable principle.

1. A proof via GLA.

It suffices to establish that IER is not derivable in GLA_\emptyset . For this we will use an appropriate Kripke model. Take

$W = \{1, 2\}$, $1 \prec 2$, P is false at 1 and 2, $\mathcal{E}(t, F)$ is always false.

$$\begin{array}{rcl} 2 & \neg P, \neg x:P, \Box x:P, \neg(\Box x:P \rightarrow P) & \text{(i.e., } \neg \text{IER)} \\ \uparrow & & \\ 1 & \neg P, \neg x:P, \neg \Box x:P, \Box x:P \rightarrow x:P & \text{(IER-soundness)} \end{array}$$

Therefore, IER is false at node 2 of the model.

2. An arithmetical proof.

If $P = \perp$, then $x:P$ is provably equivalent to \perp . Therefore, this instance of *IER* is equivalent to $\Box\perp \rightarrow \perp$, which is the consistency statement, not provable in PA.

For other reflection principles of PA see our paper [22].

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